SOLUTION OF A PROBLEM.

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PROBLEM.—If we put

$$nQ_n^{(1)} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} \dots \quad (1)$$

$$nQ_n^{(2)} = 1 + Q_2^{(1)} + Q_3^{(1)} + Q_4^{(1)} + \dots + Q_n^{(1)} \dots \quad (2)$$

$$nQ_n^{(3)} = 1 + Q_2^{(2)} + Q_3^{(2)} + Q_4^{(2)} + \dots + Q_n^{(2)} \dots \quad (3)$$

$$\dots\dots\dots\dots\dots\dots$$

$$nQ_n^{(p)} = 1 + Q_2^{(p-1)} + Q_3^{(p-1)} + \dots + Q_n^{(p-1)} \dots \quad (p)$$

$$S_n^{(p)} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \dots + \frac{1}{n^p}; \dots \quad (4)$$

then

$$Q_n^{(p-1)} = 1 - \frac{(n-1)}{1.2^p} + \frac{(n-1)(n-2)}{1.2.3^p} - \frac{(n-1)(n-2)(n-3)}{1.2.3.4^p} + \dots + (-1)^{n-1}\frac{1}{n^p} \dots (5)$$

and

$$S_n^{(p)} = n - \frac{n(n-1)}{1.2}Q_2^{(p-1)} + \frac{n(n-1)(n-2)}{1.2.3}Q_3^{(p-1)} - \frac{n(n-1)(n-2)(n-3)}{1.2.3.4} \times Q_4^{(p-1)} + \dots + (-1)^{n-1}Q_n^{(p-1)} \dots (6)$$

SOLUTION.—Take the series

$$1+x+x^2+x^3+\dots+x^{n-1} = \frac{x^n-1}{x-1} = \frac{(1+y)^n-1}{y}$$

$$= n + \frac{n(n-1)}{1.2}y + \frac{n(n-1)(n-2)}{1.2.3}y^2 + \dots + y^{n-1} \dots \quad (7)$$

in which $x = 1+y$, which gives $dx = dy$. Now multiply (7) by $dx = dy$ and integrate, then

$$x + \tfrac{1}{2}x^2 + \tfrac{1}{3}x^3 + \dots + \frac{1}{n}x^n = C + N_1y + \tfrac{1}{2}N_2y^2 + \tfrac{1}{3}N_3y^3 + \dots + \frac{1}{n}N_ny^n \dots (8)$$

$$1 + \tfrac{1}{2}x + \dots + \frac{1}{n}x^{n-1} = \frac{1}{1+y}\left(C + N_1y + \tfrac{1}{2}N_2y^2 + \dots\right)$$

$$= C(1-y+y^2-) + N_1(y-y^2+y^3) + \tfrac{1}{2}N_2(y^2-y^3+.) + + \frac{1}{n}N_n(y^n-y^{n+1}+..)$$

$$= C-(C-N_1)y + (C-N_1+\tfrac{1}{2}N_2)y^2 - (C-N_1+\tfrac{1}{2}N_2-\tfrac{1}{3}N_3)y^3 + \dots + (-1)^n(C-N_1+\tfrac{1}{2}N_2-\tfrac{1}{3}N_3+\dots+(-1)^n\frac{1}{n}N_n)y^n$$

$$= A_1 - 2A_2y + 3A_3y^2 - \dots + (-1)^{n-1}nA_ny^{n-1}.$$

In these equations $N_1 = n,\ N_2 = \frac{n(n-1)}{1.2},\ N_3 = \frac{n(n-1)(n-2)}{1.2.3}$, &c;

$$A_1 = C,\ 2A_2 = C - N_1,\ 3A_3 = C - N_1 + \tfrac{1}{2}N_2,\ 4A_4 = C - N_1 + \tfrac{1}{2}N_2 - \tfrac{1}{3}N_3, \text{ \&c.} \dots (9)$$

By integrating as before we have

$$x + \frac{1}{2^2}x^2 + \dots + \frac{1}{n^2}x^n = C' + A_1 y - A_2 y^2 + A_3 y^3 - \dots + (1-)^{n-1} A_n y^n \dots (10)$$

In a precisely similar manner we shall find

$$x + \frac{1}{2^3}x^2 + \dots + \frac{1}{n^3}x^n = C'' + B_1 y - B_2 y^2 + B_3 y^3 - \dots + (-1)^{n-1} B_n y^n \dots (11)$$

$$B_1 = C',\ 2B_2 = C' - A_1,\ 3B_3 = C' - A_1 - A_2,\ 4B_4 = C' - A_1 - A_2 - A_3, \text{ \&c.}$$

$$x + \frac{1}{2^4}x^2 + \dots + \frac{1}{n^4}x^n = C''' + E_1 y - E_2 y^2 + E_3 y^3 - \dots + (-1)^{n-1} E_n y^n \dots (12)$$

$$E_1 = C'',\ 2E_2 = C'' - B_1,\ 3E_3 = C'' - B_1 - B_2, \text{ \&c.} \dots\dots (13)$$

The integrals (8), (10), (11) and (12) are to be taken between the limits $x = 1, y = 0$, and $x = 0, y = -1$. These values in (8), (10), (11) and (12) give

$$S_n^{(1)} = C = N_1 - \tfrac{1}{2}N_2 + \tfrac{1}{3}N_3 - \dots + \frac{1}{n}(-1)^{n-1} N_n \dots (14)$$

$$S_n^{(2)} = C' = A_1 + A_2 + A_3 + \dots + A_n \dots\dots (15)$$

$$S_n^{(3)} = C'' = B_1 + B_2 + B_3 + \dots + B_n \dots\dots (16)$$

$$S_n^{(4)} = C''' = E_1 + E_2 + E_3 + \dots + E_n \dots\dots (17)$$

It will now be seen that A_{n+1}, B_{n+1}, E_{n+1}, &c., are each equal to 0, so that (8), (10), (11), and (12) are each a finite series. If we eliminate A_1, A_2, &c., by means of (9) we shall find

$$\begin{aligned} C' &= C\left(1 + \frac{1}{2} + \dots + \frac{1}{n}\right) - N_1\left(\frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{n}\right) + \tfrac{1}{2}N_2\left(\frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n}\right) \dots \\ &= C^2 - N_1(C-1) + \tfrac{1}{2}N_2(C - 1 - \tfrac{1}{2}) - \tfrac{1}{3}N_3(C - 1 - \tfrac{1}{2} - \tfrac{1}{3}) + \dots \\ &= C^2 - N_1(C-1) + \tfrac{1}{2}N_2(C - 2Q_2^{(1)}) - \tfrac{1}{3}N_3(C - 3Q_3^{(1)}) + \dots \\ &= C^2 - C\left(N_1 - \tfrac{1}{2}N_2 + \tfrac{1}{3}N_3 - \dots + \frac{1}{n}(-1)^{n-1}N_n\right) + N_1 - N_2 Q_2^{(1)} + N_3 Q_3^{(1)} \dots \\ &= N_1 - N_2 Q_2^{(1)} + N_3 Q_3^{(1)} - \dots + (-1)^{n-1} N_n Q_n^{(1)} \dots\dots (18 \end{aligned}$$

by (14).

In a similar manner we can find

$$C'' = A_1 + 2A_2 Q_2^{(1)} + 3A_3 Q_3^{(1)} + \dots + nA_n Q_n^{(1)} \dots (19)$$

$$C''' = B_1 + 2B_2 Q_2^{(1)} + 3B_3 Q_3^{(1)} + \dots + nB_n Q_n^{(1)} \dots (20)$$

By eliminating A_1, A_2, &c., from (19), we shall have

$$C'' = C(1+Q_2^{(1)}+\ldots+Q_n^{(1)}) - \mathrm{N}_1(Q_2^{(1)}+Q_3^{(1)}+\ldots+Q_n^{(1)})$$
$$+\tfrac{1}{2}\mathrm{N}_2(Q_3^{(1)}+Q_4^{(1)}+\ldots+Q_n^{(1)}) - \tfrac{1}{3}\mathrm{N}_3(Q_4^{(1)}+\ldots+Q_n^{(1)})+\ldots$$
$$= [\text{by } (2)]\ nCQ_n^{(2)} - \mathrm{N}_1(nQ_n^{(2)}-1)+\tfrac{1}{2}\mathrm{N}_2(nQ_n^{(2)}-2Q_2^{(2)})-\ldots$$
$$= nQ_n^{(2)}\left(C-\mathrm{N}_1+\tfrac{1}{2}\mathrm{N}_2-\ldots+\frac{1}{n}(-)^n\mathrm{N}_n\right)+\mathrm{N}_1-\mathrm{N}_2Q_2^{(2)}+\mathrm{N}_3Q_3^{(2)}-\ldots$$
$$= \mathrm{N}_1-\mathrm{N}_2Q_2^{(2)}+\mathrm{N}_3Q_3^{(2)}-\ldots+(-1)^{n-1}\mathrm{N}_nQ_n^{(2)}. \quad \ldots\ldots\ldots (21)$$

In a similar manner we should find

$$C''' = \mathrm{N}_1-\mathrm{N}_2Q_2^{(3)}+\mathrm{N}_3Q_3^{(3)}-\ldots+(-1)^{n-1}\mathrm{N}_nQ_n^{(3)}; \quad \ldots\ldots\ldots (22)$$

and so on for higher values of p. If we restore the values of N_1, N_2, &c., we shall thus have proved the truth of equation (6).

Now to find the value of $Q_n^{(p)}$ we have

$$nQ_n^{(1)} = \mathrm{N}_1-\tfrac{1}{2}\mathrm{N}_2+\tfrac{1}{3}\mathrm{N}_3-\ldots+\frac{1}{n}(-1)^{n-1}\mathrm{N}_n$$
$$= n-\frac{n(n-1)}{1.2^2}+\frac{n(n-1)(n-2)}{1.2.3^2}-\ldots+\frac{1}{n}(-)^{n-1}$$
$$\therefore \quad Q_n^{(1)} = 1-\frac{(n-1)}{1.2^2}+\frac{(n-1)(n-2)}{1.2.3^2}-\ldots+\frac{1}{n^2}(-1)^{n-1} \ldots (23)$$

From this we find

$$1 = 1$$
$$Q_2^{(1)} = 1-\frac{1}{1\,.\,2^2}$$
$$Q_3^{(1)} = 1-\frac{2}{1\,.\,2^2}+\frac{1\,.\,2}{1\,.\,2\,.\,3^2}$$
$$Q_4^{(1)} = 1-\frac{3}{1\,.\,2^2}+\frac{2\,.\,3}{1\,.\,2\,.\,3^2}-\frac{1\,.\,2\,.\,3}{1\,.\,2\,.\,3\,.\,4^2}$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$Q_n^{(1)} = 1-\frac{n-1}{1\,.\,2^2}+\frac{(n-1)(n-2)}{1\,.\,2\,.\,3^2}-\frac{(n-1)(n-2)(n-3)}{1\,.\,2\,.\,3\,.\,4^2}\ldots+\frac{1}{n^2}(-1)^{n-1}$$

If we sum these perpendicularly we shall find

$$nQ_n^{(2)} = n-\frac{n(n-1)}{1\,.\,2^3}+\frac{n(n-1)(n-2)}{1\,.\,2\,.\,3^3}-\ldots+\frac{1}{n^2}(-1)^{n-1} \ldots (24)$$
$$\therefore \quad Q_n^{(2)} = 1-\frac{n-1}{1\,.\,2^3}+\frac{(n-1)(n-2)}{1\,.\,2\,.\,3^3}-\ldots+\frac{1}{n^3}(-1)^{n-1} \ldots (25)$$

It is now plain that we should find in a precisely similar manner

$$Q_n^{(3)} = 1-\frac{n-1}{1\,.\,2^4}+\frac{(n-1)(n-2)}{1\,.\,2\,.\,3^4}-\ldots+\frac{1}{n^4}(-1)^{n-1}; \ldots (26)$$

and so on for higher values of p, n and p being positive integers.

The perpendicular columns above are summed by the formula

$$\Sigma[r(r+1)(r+2)\ldots(r+x-1)]=\frac{r(r+1)(r+2)\ldots(r+x)-(r-y)(r-y+1)\ldots(r+1)r}{x+1}$$

We have thus given a complete solution of the problem, and found the sum $S_n^{(p)}$ in finite terms.

THE ROTATION OF SATURN.

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THE interesting and valuable article by Professor Asaph Hall has naturally engaged the attention of readers of the ANALYST.

That Sir John Herschel should in the ninth edition of his *Outlines* give the time of rotation of Jupiter as that of Saturn, may be attributed to an accident. But in the English edition of 1849, which may perhaps be called the 2nd, if that published as part 43 of the Cabinet Cyclopædia be called the 1st, the time of rotation is stated to be $10^h\ 29^m\ 17^s$.

In the Cabinet Cyclopædia edition, at least as republished in America, no time is given.*

Grant in his History of Physical Astronomy, page 252, gives the time as $10^h\ 16^m\ 0^s.44$, derived from one hundred revolutions of the planet: this period was announced by Sir William Herschel in the Philosophical Transactions for 1794, page 62.

It is then extremely singular that Sir John Herschel should in his own first elaborate English edition of 1849 alter the time as announced by his father. Prof. Hall's conjecture seems very plausible: yet why should Sir John refer to the *Systeme du Monde* in preference to his father's original observations?

Prof. Hall having by his computations made such a comparison possible, we may be permitted to see what result would follow from a simple proportion.

Take 1st the Washington observations of December 7th $6^h\ 18^m$, and that of December 19th at $5^h\ 6^m$; the interval is one of $286^h.8$ and estimating by

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*I have discovered that the edition of Herschel's *Treatise* on astron. by S. C. Walker 1836, taken I believe from No. 24, *Cabinet Cyclopædia, does* contain the time of rotation of Saturn 10h 29m 17s.

Singularly enough, Humboldt, Cosmos Vol. 4, page 170, *in the text*, gives 10h 29m 17s, and then says, *in a note*; "the earliest and careful observations of William Herschel in November 1793, gave for Saturn's period of rotation 10h 16m 44s."